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From the Ehrenfest model to time-fractional stochastic processes

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ABSTRACT

The Ehrenfest model is considered as a good example of a Markov chain. I prove in this paper that the time-fractional diffusion process with drift towards the origin, is a natural generalization of the modified Ehrenfest model. The corresponding equation of evolution is a linear partial pseudo-differential equation with fractional derivatives in time, the orders lying between 0 and 1. I focus on finding a precise explicit analytical solution to this equation depending on the interval of the time. The stationary solution of this model is also analytically and numerically calculated. Then I prove that the difference between the discrete approximate solution at time t_n, $\forall n \geq 0$, and the stationary solution obeys a power law with exponent between 0 and 1. The reversibility property is discussed for the Ehrenfest model and its fractional version with a new observation.

1. Introduction

The Ehrenfest urn model treats a wide class of stochastic processes [1,39,2]. These processes are reversible processes, i.e. when the direction of time is reversed, the behavior of the process remains the same [3]. The partial differential equation approximated by the Ehrenfest model is a special case of the Fokker Planck equation. Many methods of solution and applications of it can be found in [4]. It can also be described as a diffusion equation with a central linear drift towards the origin [5,6]. Smoluchowski [7] showed that this equation describes also the so called Ornstein–Uhlenbeck process [8]. In recent years fractional differential equations have been studied by many mathematicians, physicists and engineers, see for example [9–11], and applied in an increasing number of fields such as physics, chemistry, signal processing [12,13], control engineering, electromagnetism, fluid mechanics [14], and finance [15]. Moving from the classical Ehrenfest model to the time fractional diffusion equation with central linear drift, also called the time fractional Fokker–Planck equation (FFPE), is a nice example of passing from a discrete to a continuous model. Fractional in time means that the first-order time derivative is replaced by the Caputo derivative of order $\beta \in (0, 1]$. This generalization describes many stochastic processes [16–18]. It interprets also the subdiffusion behavior of a particle under the combined influence of external non linear field [19]. Many attempts have been made to find an explicit solution of this time-fractional partial differential equation, see for example [20–22]. The effect of fractal external force on the asymptotic behavior of the solution is also studied in [23].

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This paper is organized as follows: In Section 2, I give the analytical solution of the time-fractional partial differential equation in a new expression depending on the interval of the time. In Section 3, the discrete scheme and its relation to the Ehrenfest model is discussed. In Section 4, the stationary solution and henceforth the reversibility property will be discussed. In Section 5, the numerical results will be displayed.

2. Solution of the time-fractional diffusion equation with drift

The partial differential equation which describes the elastic diffusive motion of a bound particle (for example a small pendulum) is a special case of the Fokker–Planck equation

$$\frac{\partial u(x,t)}{\partial t} = a\frac{\partial^2 u(x,t)}{\partial x^2} + b\frac{\partial (xu(x,t))}{\partial x}, \quad a>0,\ b>0,\ -\infty<x<\infty,\ t\geq 0, \tag{2.1}$$

here a is the diffusion constant and bx is the drift term, and if $b=0$, we have the classical diffusion equation. The conditions imposed on the solution $u(x,t)$ are $u(x,t)\geq 0$ and $\int_{-\infty}^{\infty} u(x,t) = 1$. With the initial condition $u(x,0)=\delta(x-x_0)$, the solution of Eq. (2.1) reads

$$u(x,t) = p(x_0;x,t) = \frac{1}{\sqrt{2\pi\frac{a}{b}(1-\mathrm{e}^{-2bt})}}\mathrm{e}^{\frac{-(x-x_0\mathrm{e}^{-bt})^2}{\frac{a}{b}(1-\mathrm{e}^{-2bt})}},$$

see [2,24]. This stochastic process described by Eq. (2.1) is a *Markov process*. Now I will replace the time derivative $\partial/\partial t$ by the Riemann–Liouville fractional derivative operator of order β, where $0<\beta<1$ see [25,10,26], $\underset{t}{D}^{\beta}$, and get

$$\underset{t}{D}^{\beta} u(x,t) = K_\beta L_{fp} u(x,t) + \frac{t^{-\beta}\delta(x)}{\Gamma(1-\beta)}, \tag{2.2}$$

where K_β is the diffusion constant. If $\beta=1$, one gets Eq. (2.1). It is worth saying that Eq. (2.2) is a special kind of the time-fractional Fokker–Planck equation *FFPE*, see [21,17,22], and L_{fp} is called the Fokker–Plank operator, see [4],

$$L_{fp}u(x,t) = \frac{\partial^2 u(x,t)}{\partial x^2} - \frac{\partial F(x)u(x,t)}{\partial x}. \tag{2.3}$$

Here, $F(x)$ must be an attractive linear force [21,17,22]. One can use the alternative time-fractional derivative, see [11,10], namely the Caputo time fractional derivative which is related to Riemann–Liouville by the relation

$$\underset{t*}{D}^{\beta} = \underset{t}{D}^{\beta} - \frac{t^{-\beta}\delta(x)}{\Gamma(1-\beta)}, \quad 0<\beta<1.$$

Then Eq. (2.2) is rewritten by using the Caputo fractional derivative operator of order β as

$$\underset{t*}{D}^{\beta} u(x,t) = K_\beta L_{fp} u(x,t). \tag{2.4}$$

Another version of Eq. (2.4) can be found at [27], which I call the *time-fractional diffusion equation with central linear drift towards the origin*, and is also called the *time-fractional Uhlenbeck–Ornstein process*, see [8,28], if one use the boundary conditions $u(\pm\infty,t)=0$. Clearly, the solution of Eq. (2.2) is also a solution to Eq. (2.4) and all its versions, if one uses the same boundary conditions and the same choice of $F(x)$. Therefore, let $F(x) = -\frac{ax}{K_\beta}$, and solve Eq. (2.2) by using the method of separation of variables for which I define the new independent variables

$$\tilde{x} = \sqrt{\frac{a}{K_\beta}}x, \qquad \tilde{t} = a^{-(\beta+1)}t,$$

then, write $u(\tilde{x},\tilde{t}) = X(\tilde{x})T(\tilde{t})$. Now, after using the method of separation of variables, one gets

$$\frac{\mathrm{d}^2X}{\mathrm{d}\tilde{x}^2} + \tilde{x}\frac{\mathrm{d}X}{\mathrm{d}\tilde{x}} + (n+1)X = 0, \tag{2.5}$$

and

$$\frac{\partial^\beta T}{\partial \tilde{t}^\beta} - \frac{\tilde{t}^{-\beta}}{\Gamma(1-\beta)} + nT = 0. \tag{2.6}$$

The solution of Eq. (2.6) is the Mittag–Leffler function, see [29,30],

$$T(\tilde{t}) = E_\beta(-n\tilde{t}^\beta) = \sum_{k=0}^{\infty}\frac{(-n)^k\tilde{t}^{\beta k}}{\Gamma(\beta k+1)}. \tag{2.7}$$

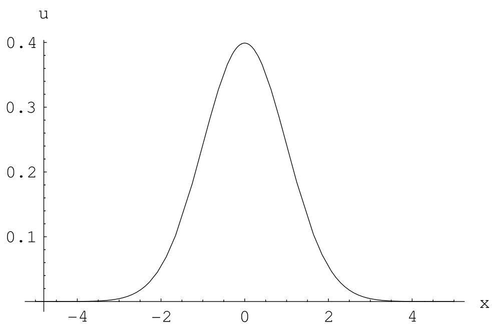

Fig. 1. The stationary analytical solution $u(x)$.

The infinite series in Eq. (2.7) exhibits a behavior similar to that of a stretched exponential for $0 < \beta < 1$ and for small values of t

$$E_\beta(-n\tilde{t}^\beta) \approxeq 1 - \frac{n\tilde{t}^\beta}{\Gamma(\beta+1)} \approxeq \exp\{-n\tilde{t}^\beta/\Gamma(\beta+1)\}, \quad 0 \le \tilde{t} \le 1. \tag{2.8}$$

Where for large t, it has the asymptotic representation

$$E_\beta(-n\tilde{t}^\beta) \sim \frac{\sin(\beta\pi)}{\pi}\frac{\Gamma(\beta)}{n\tilde{t}^\beta}, \quad t \to \infty. \tag{2.9}$$

Clearly as $\beta = 1$, $E_\beta(-\tilde{t}^\beta) \to \mathrm{e}^{-\tilde{t}}$. The solution of Eq. (2.5) is the Weber function $D_n(\tilde{x})$ of order n, see [2,24], which means

$$X(\tilde{x}) = A_n D_n(\tilde{x})\mathrm{e}^{-\tilde{x}^2/4}.$$

So far, the solution of Eq. (2.2) is obtained by summing over all n, and replacing $(\tilde{x}, \tilde{t})$ by (x, t), one gets

$$u(x,t) = \sum_{n=0}^{\infty} A_n E_\beta(-nt^\beta) D_n(x)\mathrm{e}^{-x^2/4}, \tag{2.10}$$

where A_n is a constant obtained by the initial conditions. Suppose that $p(y, 0) = h(y)$, then

$$A_n = \frac{1}{n!\sqrt{2\pi}} \int_{-\infty}^{\infty} h(y) D_n(y)\mathrm{e}^{y^2/4}\mathrm{d}y.$$

Now, we can find an explicit formula to $u(x, t)$, Eq. (2.10), **only for small values of** t. First, suppose that $\acute{t} = t^\beta/\Gamma(\beta+1)$, then the stretched exponential, Eq. (2.8), tends to $\mathrm{Exp}(-n\acute{t})$. Second, use the well known property of the Weber function

$$\sum_{n=0}^{\infty} \frac{D_n(y)D_n(x)\mathrm{e}^{-n\acute{t}}}{n!} = \frac{\mathrm{e}^{(y^2+x^2)/4}}{(1-\mathrm{e}^{-2\acute{t}})^{1/2}} \exp\left\{\frac{-(y^2+x^2-2xy\mathrm{e}^{-\acute{t}})}{2(1-\mathrm{e}^{-2\acute{t}})}\right\}.$$

Third, use $h(y) = \delta(y - x_0)$. Finally substitute all these formulae on Eq. (2.10) and integrate over y from $-\infty \to \infty$, to get

$$u(x,t) = \frac{1}{\sqrt{2\pi(1-\mathrm{e}^{-2t^\beta/\Gamma(\beta+1)})}} \exp\left\{\frac{-(x - x_0\mathrm{e}^{-t^\beta/\Gamma(\beta+1)})^2}{2(1-\mathrm{e}^{-2t^\beta/\Gamma(\beta+1)})}\right\}. \tag{2.11}$$

This solution is valid for $-\infty < x < \infty$ and only for $0 \le t \le 1$. Clearly as $\beta = 1$, Eq. (2.11) gives the same solution of Uhlenbeck–Ornstein process, see [28,27]. Now, The stationary solution can be found from all these expressions of $u(x, t)$ by taking the limit as $t \to \infty$, to get

$$u(x) = \frac{1}{\sqrt{2\pi}} \exp(-(x - x_0)^2/2).$$

This stationary solution of the Ehrenfest model and its fractional generalization converges to the same gaussian density function. $u(x)$ is plotted at Fig. 1.

3. Discretization of the Ehrenfest model and its time-fractional generalization

To discretize Eq. (2.1), I utilize the common finite difference methods. Therefore one needs to adjust the spatial step h such that $R = \frac{2a}{bh^2}$ is an integer number, where $a > 0$ and $b > 0$, to get

$$x_k = kh, \quad h > 0, \quad k \in [-R, R], \quad R \in \mathbb{N}, \tag{3.1}$$

and adjust τ in order to get

$$t_n = n\tau, \quad \tau > 0,\ n \in \mathbb{N}_0. \tag{3.2}$$

Then for the probability density function $u(x, t)$, we introduce $y_j(t_n)$ as an approximation to its integral over the finite interval $[-Rh, Rh]$, see [31,24]. That means

$$y_j(t_n) \approx \int_{x_j-h/2}^{x_j+h/2} u(x, t_n)\mathrm{d}x \approx hu(x_j, t_n) \tag{3.3}$$

where $u(x, t)$ satisfies the differential equation (2.1) and is interpreted as a probability density function. Then according to Eq. (3.3), one introduces the vector

$$y^{(n)} = \{y^{(n)}_{-R}, y^{(n)}_{-R+1}, \dots, y^{(n)}_{R-1}, y^{(n)}_{R}\}^{\mathrm{T}}.$$

Here $y^{(n)} = y(t_n)$ is a probability column vector. One suitably chooses the initial value $y^{(0)}$ such that $\sum_{j=-R}^{R} y_j^{(0)} = 1$. Therefore one has $\sum_{j=-R}^{R} y_j^{(n)} = 1\ \forall n \in \mathbb{N}_0$. Discretizing by symmetric differences in space forward in time, we generate a discrete solution to our model, i.e. to the diffusion with central linear drift $-bx$ described by Eq. (2.1)

$$\frac{y_j^{(n+1)} - y_j^{(n)}}{\tau} = a\frac{y^{(n)}_{j+1} - 2y^{(n)}_j + y^{(n)}_{j-1}}{h^2} + \frac{b}{2h}\left(x_{j+1}y^{(n)}_{j+1} - x_{j-1}y^{(n)}_{j-1}\right) \tag{3.4}$$

the scaling relation

$$\mu = \frac{\tau}{h^2}, \quad 0 < \mu \le \frac{1}{2}. \tag{3.5}$$

Therefore for the grid points $x_j = jh$, the index j must be restricted to the range $\{-R, -R+1, \dots, R-1, R\}$ where $R \in \mathbb{N}$. We complement Eq. (3.4) by prescribing the non-negative initial value $y^{(0)}$ obeying $\sum_{j=-R}^{R} y_j^{(0)} = 1$, and for convenience $y_j^{(0)} = 0 \forall |j| \ge R+1$. It is now worth saying that Eq. (2.1) with $\frac{\tau}{h^2} = \frac{1}{2}, R = \frac{2a}{bh^2}$ can be rewritten in the form of Eq. (3.4). With symmetric difference quotients in space, we have consistency of this approximation scheme of order $(\tau + h^2)$ for $(h \to 0$ and $\tau \to 0)$ to the partial differential equation (2.1), where $R = \frac{2a}{bh^2} \to \infty$, so that in the limit the whole real axis is covered for the variable x. Now solving Eq. (3.4) for $y_j^{(n+1)}$, we get

$$y_j^{(n+1)} = (1 - 2a\mu)y_j^{(n)} + \mu\left(a + \frac{j+1}{R}\right)y^{(n)}_{j+1} + \mu\left(a - \frac{j-1}{R}\right)y^{(n)}_{j-1}. \tag{3.6}$$

Eq. (3.6) is equivalent to

$$y_j^{(n+1)} = \gamma y_j^{(n)} + \lambda_{j+1}y^{(n)}_{j+1} + \rho_{j-1}y^{(n)}_{j-1}, \quad -R \le j \le R, \tag{3.7}$$

with

$$\rho_j = p_{j,j+1} = \mu\left(a - \frac{j}{R}\right), \quad \gamma = p_{j,j} = (1 - 2a\mu) \quad \text{and} \quad \lambda_j = p_{j,j-1} = \mu\left(a + \frac{j}{R}\right).$$

These transition probabilities satisfy the essential condition

$$\rho_j + \lambda_j + \gamma = 1 \quad \forall j \in [-R, R],$$

where $\rho_{-R-1} = \lambda_{R+1} = 0$. With these conditions, the transition probabilities λ_j, γ and ρ_j constitute a tridiagonal matrix $P = (p_{i,j})$, where $p_{i,j} = 0 \forall |i - j| \ge 2$. For ease of writing, we henceforth take

$$a = 1, \qquad b = 1. \tag{3.8}$$

This simplification is allowed because the change of variables

$$x = \alpha\acute{x}, \qquad t = \beta\acute{t}, \qquad u(x, t) = v(\acute{x}, \acute{t}),$$

transforms Eq. (2.1) into

$$\frac{\partial v}{\partial \acute{t}} = \frac{a\beta}{\alpha^2}\frac{\partial^2 v}{\partial \acute{x}^2} + b\beta\frac{\partial}{\partial x}(\acute{x}v), \tag{3.9}$$

and with $\beta = 1/b$, $\alpha = \sqrt{a/b}$, we get $\frac{a\beta}{\alpha^2} = 1$, $b\beta = 1$. Then the matrix P takes the form

$$P = \begin{pmatrix} (1-2\mu) & 2\mu & 0 & 0 & 0 & \dots & 0 \\ \frac{\mu}{R} & (1-2\mu) & \mu\left(2-\frac{1}{R}\right) & 0 & 0 & \dots & 0 \\ 0 & \frac{2\mu}{R} & (1-2\mu) & \mu\left(2-\frac{2}{R}\right) & 0 & \dots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \dots & 0 & \mu\left(2-\frac{2}{R}\right) & (1-2\mu) & \frac{2\mu}{R} & 0 \\ 0 & \dots & 0 & 0 & \mu\left(2-\frac{1}{R}\right) & (1-2\mu) & \frac{\mu}{R} \\ 0 & \dots & 0 & 0 & 0 & 2\mu & (1-2\mu) \end{pmatrix}.$$

Since

$$p_{i,j} \geq 0, \quad \text{and} \quad \sum_{j=-R}^{R} p_{i,j} = 1 \quad \forall i,$$

the matrix P is a stochastic matrix and it represents the transition matrix of a *Markov chain*. For the interpretation of $y^{(n)}$ as a vector of probabilities, we need the condition of preservation of non-negativity which requires that

$$\mu \leq 1/2 \quad \text{and} \quad -R \leq j \leq R.$$

We interpret Eq. (3.7) as the probability of finding the particle at the point x_j at the time instant t_{n+1}, where at the previous time instant t_n the particle may be at the point x_{j-1}, x_j or x_{j+1}. In order to have transition probabilities from the time instant t_n to t_{n+1}, it is required that $\gamma \geq 0$ and all the occurrences of $\lambda_j \geq 0$ and $\rho_j \geq 0$, hence $0 < \mu \leq \frac{1}{2a}$ are positive numbers. Besides the interpretation of Eq. (3.7) as a random walk, it can also be considered as a generalized Ehrenfest urn model, see Vincze [32,8,33,5,34,35,6].

Now, I want to show the relation between the Ehrenfest model and the time-fractional diffusion equation with drift Eq. (2.4). Therefore, one needs to find first the discretization of Eq. (2.4). To do so, the backward Grünwald–Letnikov scheme is utilized in time (starting at level $t = t_{n+1}$) to discretize the Caputo time-fractional derivative by the expression

$$\sum_{k=0}^{n+1} (-1)^k \binom{\beta}{k} \frac{y_j^{(n+1-k)} - y_j^{(0)}}{\tau^\beta}, \quad 0 < \beta \leq 1. \tag{3.10}$$

Observe that for $\beta = 1$, this reduces to

$$\underset{\tau\,*}{D}{}^1 y_j^{(n+1)} = \frac{1}{\tau}\left(y_j^{(n+1)} - y_j^{(n)}\right).$$

Note that in case of sufficient smoothness, the scheme (3.10) has order of accuracy $O(h^2 + \tau)$. Now, Solving Eq. (3.10) for $y_j^{(n+1)}$, $-R \leq j \leq R$, gives

$$\begin{aligned} y_j^{(n+1)} &= \sum_{k=0}^{n} (-1)^k \binom{\beta}{k} y_j^{(0)} + \sum_{k=1}^{n} (-1)^{k+1} \binom{\beta}{k} y_j^{(n+1-k)} \\ &\quad + y_{j+1}^{(n)} \left[\mu + \frac{\mu h^2}{2}(j+1)\right] - 2\mu y_j^{(n)} + y_{j-1}^{(n)} \left[\mu - \frac{\mu h^2}{2}(j-1)\right]. \end{aligned} \tag{3.11}$$

Again $y^{(n+1)}$ represents the probability column vector for where to find the particle at the time instant t_{n+1} and it depends on $y_{j-1}^{(n)}, y_j^{(n)}, y_{j+1}^{(n)}, y_j^{(n-1)}$,. . ., and back to $y_j^{(0)}$. This means that the solution of the time-fractional diffusion equation depends on the past. The following term

$$\sum_{k=0}^{n} (-1)^k \binom{\beta}{k} y_j^{(0)} + \sum_{k=1}^{n} (-1)^{k+1} \binom{\beta}{k} y_j^{(n+1-k)}$$

represents the memory, while

$$y_{j+1}^{(n)} \left[\mu + \frac{\mu h^2}{2}(j+1)\right] - 2\mu y_j^{(n)} + y_{j-1}^{(n)} \left[\mu - \frac{\mu h^2}{2}(j-1)\right]$$

represents the diffusion.

Therefore, this model has the form of an explicit difference scheme, and can be interpreted as a random walk with a memory, see [34]. To prove that this difference scheme is conservative and non-negative with respect to its dependence on the past, i.e. $\sum_{j=1}^{n} y_j^{(0)} = 1\forall n$, see [27]. In fact, $y_j^{(n+1)}$ with $-R \le j \le R$, Eq. (3.11), constitute the column vector $y^{(n+1)}$ and is written in the matrix form

$$y^{(n+1)} = Q^{\mathrm{T}}.y^{(n)}, \tag{3.12}$$

where, the matrix $Q = (q_{i,j})$ has the form

$$Q = \begin{pmatrix} (\beta-2\mu) & 2\mu & 0 & \dots & \dots & \dots & 0 \\ \frac{\mu}{R} & (\beta-2\mu) & \mu\left(2-\frac{1}{R}\right) & 0 & \dots & \dots & 0 \\ 0 & \frac{2\mu}{R} & (\beta-2\mu) & \mu\left(2-\frac{2}{R}\right) & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & 0 & \mu\left(2-\frac{2}{R}\right) & (\beta-2\mu) & \frac{2\mu}{R} & 0 \\ \dots & \dots & \dots & 0 & \mu\left(2-\frac{1}{R}\right) & (\beta-2\mu) & \frac{\mu}{R} \\ \dots & \dots & \dots & 0 & 0 & 2\mu & (\beta-2\mu) \end{pmatrix}.$$

This matrix is not a stochastic matrix because $\sum_{j=-R}^{R} q_{i,j} = \beta\forall i, n \ge 1$, in fact it is stochastic, only at $n = 0$. It means the transition from the time instant t_n to t_{n+1} is not Markov-like for $n \ge 1$ and is Markov only at $n = 0$, see [1].

4. Convergence and reversibility

In order to discuss the convergence of the solution of the time-fractional diffusion equation with drift, one has to find first the stationary solution. To do so, it is convenient to write the matrix Q as

$$Q = \beta I + \mu H, \tag{4.1}$$

where I is a unit matrix and H is a square matrix whose rows sum to zero and is defined as, see [36],

$$H = \begin{pmatrix} -2 & 2 & 0 & 0 & \dots & \dots & \dots & 0 \\ \frac{1}{R} & -2 & \left(2-\frac{1}{R}\right) & 0 & \dots & \dots & \dots & 0 \\ 0 & \frac{2}{R} & -2 & \left(2-\frac{2}{R}\right) & 0 & \dots & \dots & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & \dots & \dots & 0 & \left(2-\frac{2}{R}\right) & -2 & \frac{2}{R} & 0 \\ 0 & \dots & \dots & 0 & 0 & \left(2-\frac{1}{R}\right) & -2 & \frac{1}{R} \\ 0 & \dots & \dots & 0 & 0 & 0 & 2 & -2 \end{pmatrix}. \tag{4.2}$$

This matrix does not depend on β, and it has an eigenvector y^* with eigenvalue zero. Define $\bar{y} = cy^*$, to be the stationary solution of the matrix Q, where $c = 1/\sum_{j=-R}^{R} y_j^*$ and $\sum_{-R}^{R} \bar{y}_j = 1$. Actually Vincze, Fritz et al. and Kac, see [32,37], showed that the elements of the stochastic matrix P converge to the binomial distribution at $n \to \infty$ as

$$\lim_{n\to\infty} P^n = \begin{pmatrix} \pi_0 & \pi_1 & \dots & \pi_N \end{pmatrix},$$

where

$$\pi_j = 2^{-2R} \binom{2R}{j+R}, \tag{4.3}$$

and $\sum_{-R}^{R} \pi_j = 1$. The numerical results show that $\bar{y}$ and π are identical column vectors, see Figs. 2 and 3. This means that the approximate solution of the time-fractional diffusion equation with central linear drift is the same as the stationary solution of the Ehrenfest model. as $t \to \infty$, i.e. both models have the same stationary solution and as $t \to \infty$, β plays no role.

Now, I want to find the convergence of the discrete approximate solution of both models. Define, first the vector z such that $z^{(n)} = (y^{(n)})^{\mathrm{T}}$, and the difference vector $d(t) = \{d(t_0), d(t_1), d(t_2), \dots\}$, where

$$d(t_i) = \sum_{j=-R}^{R} |z_j^{(n)} - \pi_j|, \quad i = 0, 1, \dots. \tag{4.4}$$

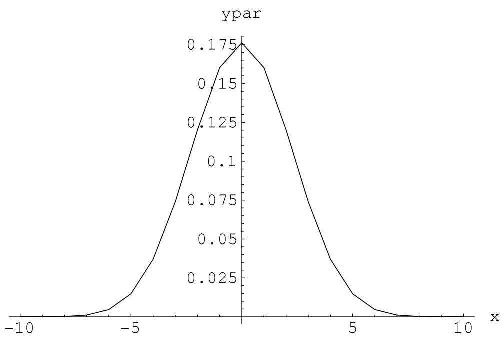

Fig. 2. $\bar{y}$.

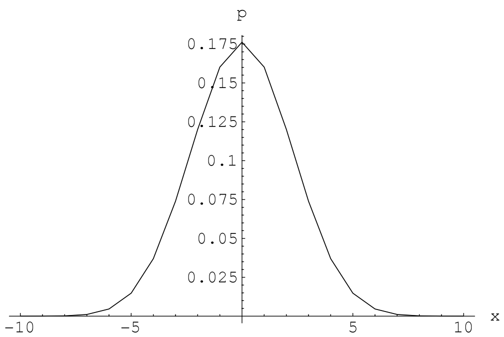

Fig. 3. π.

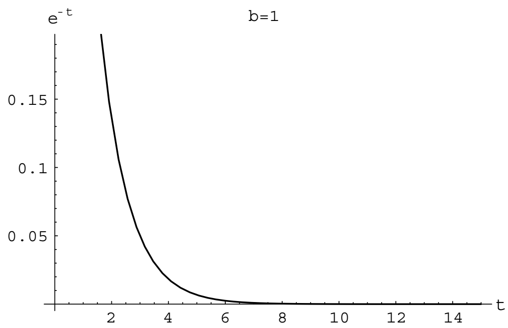

Fig. 4. $\exp(-t), t : 0 \to 15$.

The numerical results show that for the Ehrenfest model, i.e. $\beta = 1$, the row vector $d(t)$ approximates an exponential function of the form

$$d(t) \approx c\mathrm{e}^{-\omega t},$$

where ω and c are constants and ω is called the rate of convergence, see Fig. 12, while for $\beta \in (0, 1)$, the simulation shows that the row vector d approximates a power function of the form

$$d(t) \approx ct^{-\gamma},$$

where c and γ are constants and γ is also called the rate of convergence, see Fig. 13. The behavior of $d(t)$ is exponential as $\beta = 1$ and it has a fat tail characterized by power law with exponent $\in (0, 1)$ as $0 \le \beta \le 1$, see Fig. 13. The decays at Figs. 12 and 13 are similar to the decays at Figs. 4 and 5 respectively, which proves my point.

The reversibility property of the classical Ehrenfest model, i.e. $s = 0$, has been proved, see [38]. The definition of the reversible process as it is stated at the book of Kelly [3] is as follows: A stochastic process $X(t)$ is said to be reversible if $X(t_1), X(t_2), \ldots, X(t_n)$ has the same distribution as $X(\tau - t_1), X(\tau - t_2), \ldots, X(\tau - t_n)$, for all $t_1, t_2, \ldots, t_n, \tau \in \mathbb{R}$, see [3]. The book states the condition of the reversibility at Theorem 1.2 that is the stationary Markov chain is reversible if and only if there exists a collection of positive numbers $\pi_j, j \in \mathbb{N}$, summing to unity and satisfying the detailed balance conditions

$$\pi_j p_{j,k} = \pi_k p_{k,j}, \quad j, k \in \mathbb{N}. \tag{4.5}$$

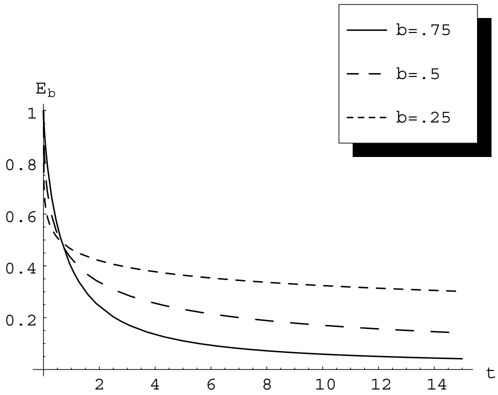

Fig. 5. $E_\beta(-t^\beta), t : 0 \to 15$.

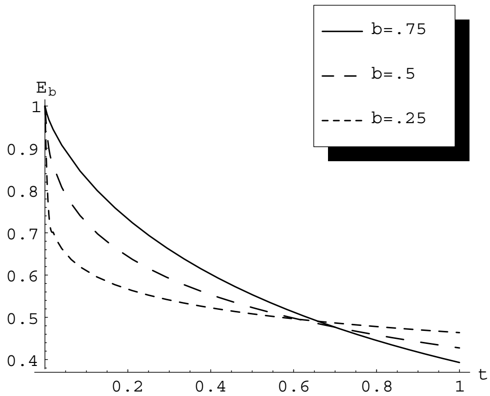

Fig. 6. $E_\beta(-t^\beta), t : 0 \to 1$.

It is known that the classical Ehrenfest model is a Markov Process, therefore, the identity (4.5) is clearly satisfied, i.e.

$$\sum \bar{y}.P = \sum P^{\mathrm{T}}.\bar{y} = \beta, \quad \forall \beta \in (0, 1].$$

By applying this theorem at the fractional diffusion processes with central linear drift. Numerical calculations show that

$$\sum \bar{y}.Q = \sum Q^{\mathrm{T}}.\bar{y} = \beta, \quad \forall \beta \in (0, 1].$$

This means, although the time-fractional diffusion with central linear drift is not a Markov type process, its stationary solution satisfies the identity (4.5). This means, as $t \to \infty$, the fractional order β plays no role, i.e. the memory part does not affect the behavior of the solution. Finally, we can conclude that as $t \to \infty$, the stationary solution of the time-fractional diffusion with central linear drift behaves as Markov-like and the process turns to be a reversible process.

5. Numerical results

The numerical approximate solutions are plotted for $R = 10$, $\mu = 0.25$, with the initial condition $y^{(0)} = \{0, \dots, 1, \dots, 0\}$. The iteration index n is calculated from the relation $n\tau = t_i$, $i = 1, 2, \dots \to \infty$, while τ is calculated from the scaling parameter (3.5), and the independent variables x and t are calculated from Eqs. (3.1) and (3.2). First, we plot the function e^{-t} at Fig. 4 to show its fast decay. Fig. 5 shows graphs of the functions $E_\beta(-t^\beta)$ for $0 \le t \le 15$. Fig. 6 exhibits the same function in the small interval (i. e. $0 \le t \le 1$). The stretched exponential function (2.9) is plotted in Fig. 7. These two figures show that the stretched exponential and the Mittag–Leffler function have the same behavior for t near zero, but as t increases the stretched exponential function decays faster than the Mittag–Leffler function. The letter b being used at the graphs refers to the time-fractional order β, see [20]. We plot the analytical solution $u(x, t)$ of Eq. (2.1) for $t : 0 \to 1$, for $\beta = 1$ at Fig. 8. The solution of Eq. (2.4), for $\beta = 0.75$ at Fig. 9, for $\beta = 0.5$ at Fig. 10, and for $\beta = 0.25$ at Fig. 11. The curve at Fig. 8 decays rapidly as t reaches 1. At Fig. 9, the curve is approximately similar to the one at Fig. 8 because $\beta = 0.75$ but the curves at Figs. 10 and 11 are wider than the curves at Fig. 8 and Fig. 9. The values of the independent coordinate x and the function $u(x, t)$ decrease as β decreases. The difference vector $d(t)$ is plotted at Fig. 12 for $\beta = 1$ and for $\beta = .5$ at Fig. 13, see Section 4.

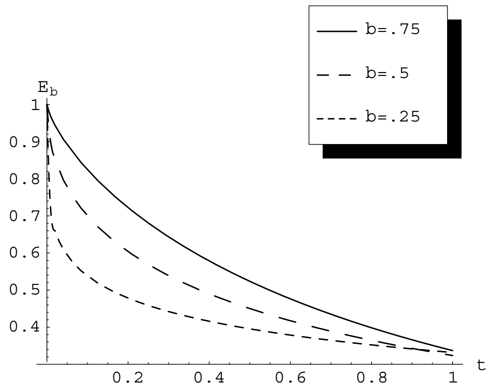

Fig. 7. $\exp(-t^{\beta/\Gamma(1+\beta)})$, $t : 0 \rightarrow 1$.

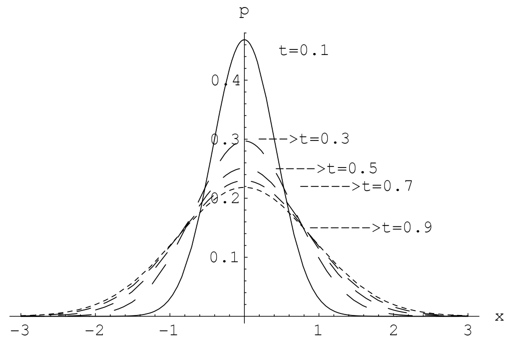

Fig. 8. $u(x, t)$, $\beta = 1$.

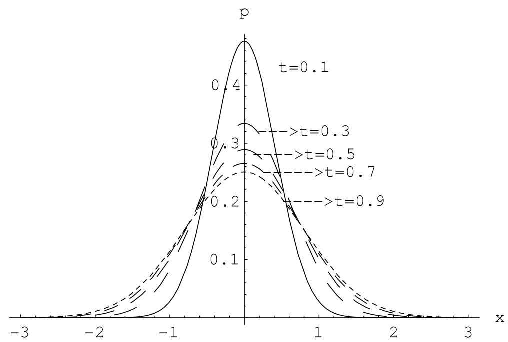

Fig. 9. $u(x, t)$, $\beta = 0.75$.

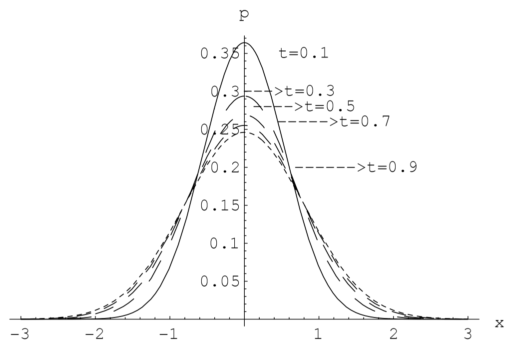

Fig. 10. $u(x, t)$, $\beta = 0.5$.

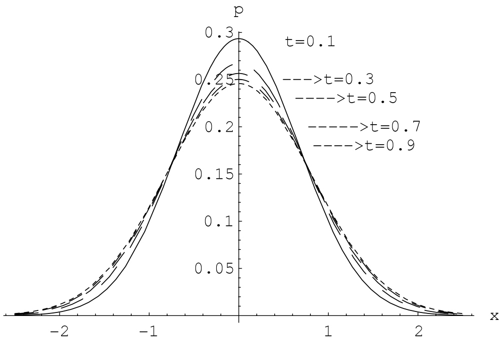

Fig. 11. $u(x, t)$, $\beta = 0.25$.

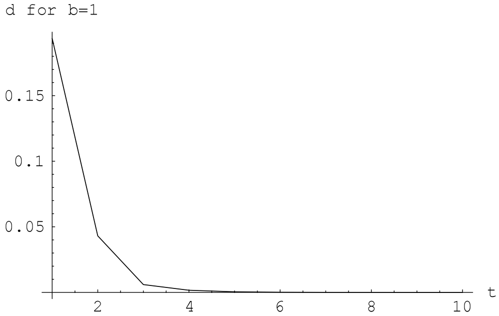

Fig. 12. Convergence for $\beta = 1$.

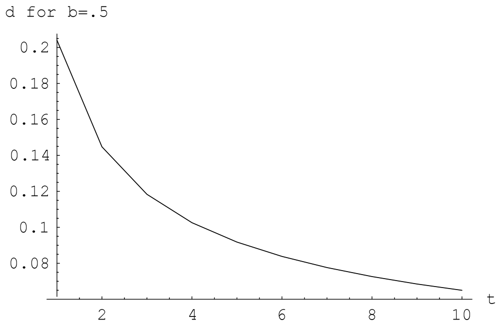

Fig. 13. Convergence for $\beta = 0.5$.

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